

CHAPTER I: TIDAL FORCES

1. BASICS OF CELESTIAL MECHANICS

Newton's Second Law states a relationship between net force and acceleration $F = ma$. For a falling object at the surface of the earth, the acceleration due to the force of gravity is usually written g , so that

$$F = mg. \quad (\text{I.1})$$

Here $g \approx 9.8 \frac{m}{s^2}$, and m is the mass of the falling object.

Newton's Law of Gravitation is an expression of the mutual gravitational attraction between two masses, m_1 and m_2 . The attraction is proportional the product of the masses of the two bodies and inversely proportional to the square of the distance separating their centers of mass (l). In the case of the falling object of mass m , and the earth, with mass M_e , we write

$$F_g = G \frac{M_e m}{l^2} \quad (\text{I.2})$$

Here G is the gravitational constant. At the earth's surface $l = r_e$, and Eqs (I.1) and (I.2) yield,

$$mg = G \frac{M_e m}{r_e^2} \quad (\text{I.3})$$

The **earth's mass** is 5.97×10^{24} kg, and the **earth's radius** is 6.371×10^6 m. Therefore, from Eq (I.3), the magnitude of the **gravitational constant** follows:

$$G = 6.673 \times 10^{-11} \left(\frac{m^3}{kg \ s^2} \right). \quad (\text{I.4})$$

The masses of the sun and moon relative to earth are

$$\text{Sun } 333420; \quad \text{Earth } 1; \quad \text{Moon } \frac{1}{81.53};$$

Kepler's First Law states that the orbit of a planet is an **ellipse** with the sun located at one focus. Fig. I.1 depicts an ellipse with the sun (S) at one of the foci. Denoting the distance $A'C = AC = a$ as the major semi-axis, the distance from the center (C) to the focus is ae . Then $e = ae/a$ is called eccentricity of the ellipse. The distance of the planet from the sun is the shortest at **perihelion**

$$SA = a - ae = a(1 - e) \quad (\text{I.5a})$$

At **aphelion** (point A') the planet is at the longest distance from the sun,

$$SA' = a + ae = a(1 + e) \quad (\text{I.5b})$$

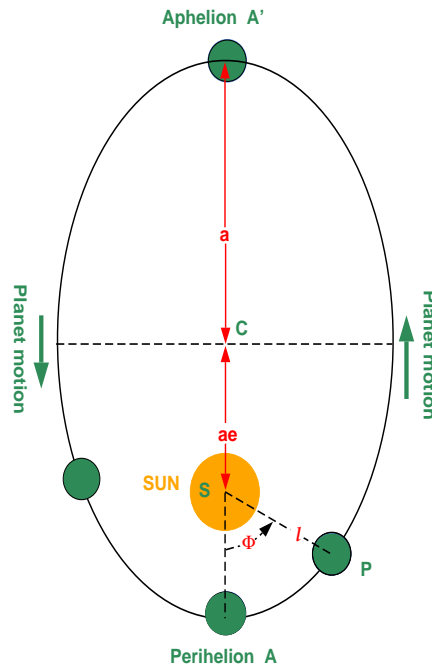


Figure I.1

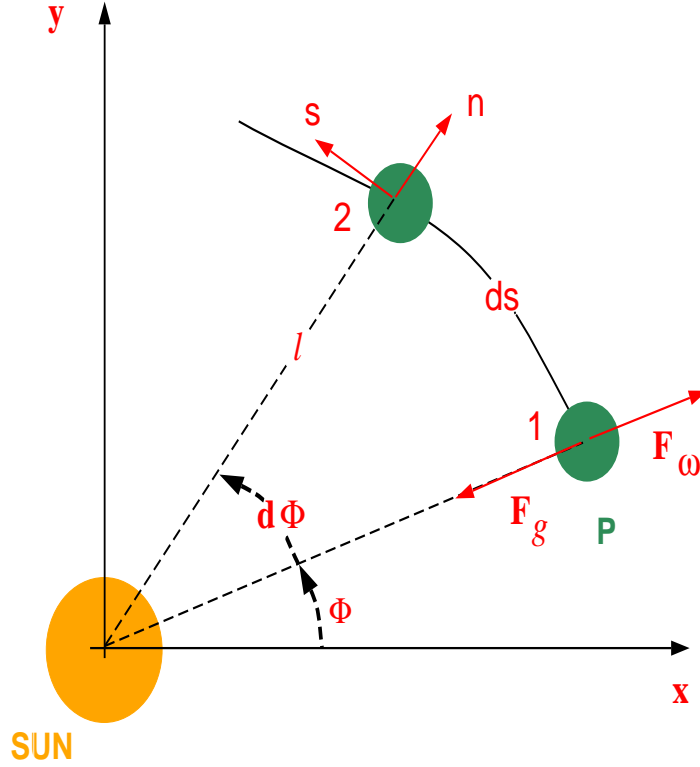
Elliptical orbits. Semi-major axis a , eccentricity $e = ae/a$, l is the distance from the center of the sun (S) to the center of the planet (P), ϕ is the true anomaly.

The measured distances in eq(I.5a) and eq(I.5b) give for the **eccentricity of the Earth's orbit** $e = 0.01674$. Equation of an ellipse can be also written in the polar coordinates. Denoting distance from the focus (S) to the planet (P) as the radius vector l and the angle ASP (from perihelion counterclockwise) as ϕ , we can write;

$$l = \frac{a(1 - e^2)}{1 + e \cos \phi} \quad (\text{I.6})$$

The angle ϕ is often called the **true anomaly** of the planet. For $\phi = 0^\circ$ (perihelion) l is equal to eq(I.5a) and for $\phi = 180^\circ$ (aphelion), l is equal to eq(I.5b).

Consider polar coordinates (l, ϕ) given in Fig. I.2. Setting the origin of coordinates at the center of the sun, the radius vector \vec{l}/l is directed from the sun to the planet. Vector \vec{s} denotes direction along the planet's path and vector \vec{n} is normal to s . To define components of velocity let the planet P travels a small distance ds along the arc from the point 1 to the point 2.

**Figure I.2**

System of coordinates for the rotating planet. Center of coordinate located at the sun. l is radius vector, ds is the element of the orbit and $d\phi$ is the angle related to ds . F_g is force of attraction and F_ω is centrifugal force.

Components of the velocity vector $\vec{v} = \frac{d\vec{l}}{dt}$ along the tangential and normal directions can be defined as,

$$v_s = l \frac{d\phi}{dt} = v \quad v_n = 0 \quad (\text{I.7a})$$

and the components of acceleration vector $\frac{d^2\vec{l}}{dt^2}$ are,

$$\frac{dv_s}{dt} = \frac{dv}{dt} \quad \frac{dv_n}{dt} = \frac{v^2}{l} \quad (\text{I.7b})$$

Setting M_s as the mass of the sun and M_p that of a planet, the force of attraction expressed by eq(I.2) can be written in the vector form as,

$$\vec{F}_g = -G \frac{M_s M_p}{l^2} \frac{\vec{l}}{l} \quad (\text{I.8})$$

Using above defined expression for the acceleration along the direction normal to the planet's arc the centrifugal force is

$$\vec{F}_\omega = M_p \frac{v^2}{l} \frac{\vec{l}}{l} \quad (\text{I.9})$$

To gain insight into the second Kepler law which states that the radius vector from the sun to the planet describes equal areas in equal times; we specify the moment of force. Introducing moment of the force \vec{N} as the vector product of the force \vec{F}_g and the radius vector

$$\vec{N} = \vec{l} \times \vec{F}_g \quad (\text{I.10})$$

it follows that $\vec{N} = 0$, since both force and radius vector are parallel.

For the planet in its motion around the sun the vector sum of the two forces ought to be zero,

$$M_p \frac{d^2 \vec{l}}{dt^2} - G \frac{M_s M_p}{l^2} \frac{\vec{l}}{l} = 0 \quad (\text{I.11})$$

Vector product of this equation by radius vector \vec{l} yields

$$M_p \vec{l} \times \frac{d^2 \vec{l}}{dt^2} = 0 \quad \text{and therefore} \quad M_p \vec{l} \times \frac{d\vec{l}}{dt} = \text{Const} \quad (\text{I.12})$$

Since the angular momentum of planet is constant it will be easy to conclude that the areal velocity is constant as well. Eq(I.12) leads to the following results

$$M_p |l| \left| l \frac{d\phi}{dt} \right| \sin(90^\circ) = M_p l^2 \frac{d\phi}{dt} = \text{Const} \quad (\text{I.13})$$

Introducing polar coordinates as in Fig. I.2, with the radius l and the polar angle ϕ , the element of surface is

$$ds = \frac{1}{2} l^2 d\phi \quad (\text{I.14})$$

or differentiating above equation with respect to time it follows from eq(I.13) that,

$$2 \frac{ds}{dt} = l^2 \frac{d\phi}{dt} = \text{Const} \quad (\text{I.15})$$

Thus the law of angular momentum leads to conclusion that planets move with the constant areal velocity. Which in turn confirm well known observations that linear velocity along the orbit of the planet is variable with the largest velocity at perihelion (shortest radius vector) and smallest at aphelion (see Fig. I.1).

2. TIDE-PRODUCING FORCES

Tides are oscillations of the ocean caused by gravitational forces of the sun and earth. Although the nature of the tide generating forces is well understood the details of the tidal phenomena in the various water bodies require knowledge of the wide range of variables including bathymetry and frictional forces. To simplify consideration we shall make assumption that the tidal phenomenon is dominated by the moon. As is well known the centers of gravity of earth and of moon are moving around the **common center of gravity (barycentre)**. Actually, this system is a twin planets system moving around the common center. Barycentre, due to the difference of the mass of earth and moon, is located approximately 3/4 of the earth's radius from the earth center. To find the position

of the common center lets use Fig. I.3 and denote the distance from the common center to the center of the earth as r_{oe} and to the center of the moon as r_{om} .

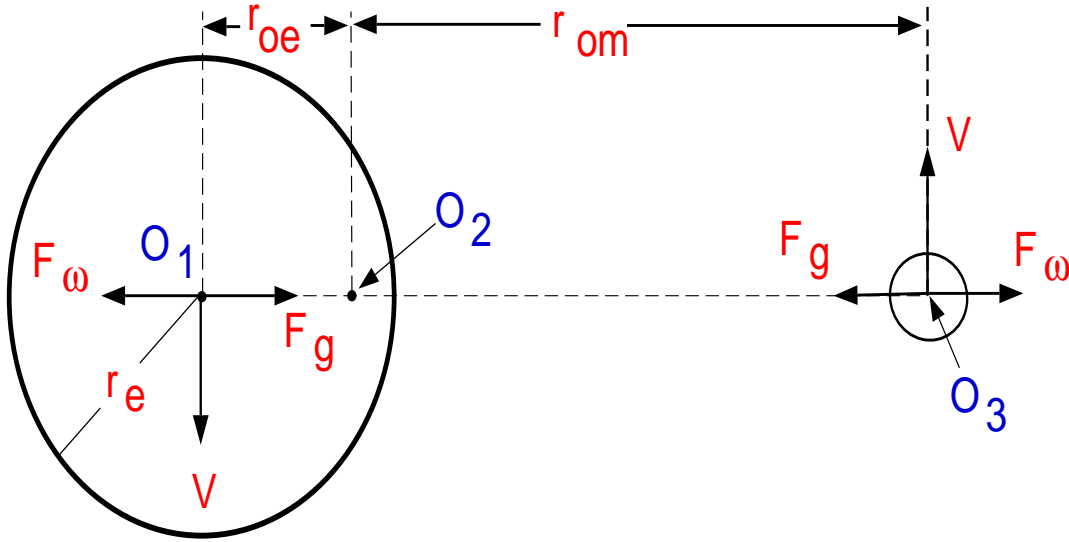


Figure I.3

Earth-moon interaction. O_2 location of the common center of mass. V tangential velocity to the trajectory.

The coordinate of the barycentre r_c are defined as,

$$r_c(M_m + M_e) = r_1M_e + r_2M_m \quad (\text{I.16})$$

Setting the origin of the coordinates at the center of the earth, yields $r_c = r_{oe}$, $r_1 = 0$, $r_2 = l = r_{oe} + r_{om}$, therefore, eq(I.16) simplifies to

$$r_{oe} = \frac{M_m l}{M_m + M_e} \quad (\text{I.17})$$

The mean distance between the center of the earth and the center of the moon is close to 60 earth radii, $l = 60.3r_e$. Since $M_e = 81.53M_m$, from eq(I.17) $r_{oe} = r_e 60/82$.

The earth and moon are locked in the motion, rotating with the period of one month. The earth-moon system is kept in the dynamical equilibrium by two forces. One of them is the centrifugal force:

$$F_\omega = M \frac{V^2}{r_o} \quad (\text{I.18})$$

Here M is the mass of earth or the moon, V is the velocity of the earth or moon and r_o is the radius of the orbit either of the earth ($r_o = r_{oe}$) or the orbit of the moon ($r_o = r_{om}$)

(see Fig. I.3). The second force (F_g) is the force of gravitational attraction. For the earth-moon system it is expressed as:

$$F_g = G \frac{M_e M_m}{l^2} \quad (\text{I.19})$$

Here $l = r_{oe} + r_{om}$ is the time-dependent distance between the center of the earth and the center of the moon. For the system earth-moon to be in the equilibrium the vector sum of the two forces ought to be zero, both in the center of the earth and in the center of the moon (see Fig. I.3). Therefore, \vec{F}_ω must be equal \vec{F}_g ,

$$\vec{F}_\omega = \vec{F}_g \quad (\text{I.20})$$

The rotational motion around common center of gravity is somewhat different from the motion described by a wheel. Earth and moon revolves around the common center without rotation through a simple translation. To explain this revolution without rotation we invoke geometrical interpretation given by [Darwin \(1901\)](#) and [Defant \(1960\)](#). Consider points A and B on the earth (Fig. I.4).

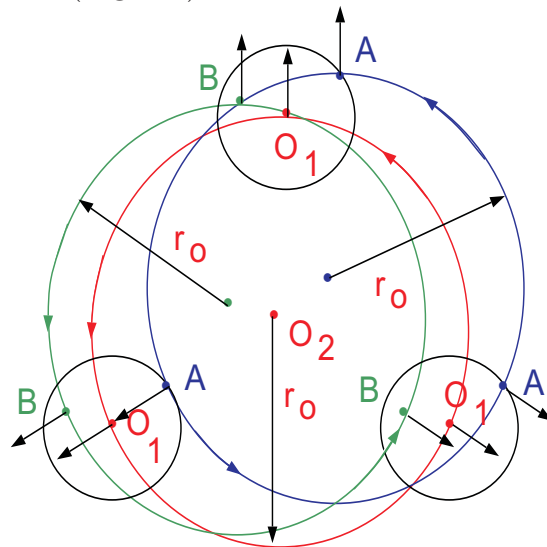


Figure I. 4

Rotation around common center O_2 . System of the centrifugal forces and circular orbits traced by points A , B , and O_1 .

To simplify considerations and to make picture more lucid we shall move the barycentre O_2 from inside the earth to the outside. The revolution of the earth around point O_2 proceeds in such way that every particle located on earth describe a circle of the same radius $r = r_o$ (Fig. I.4). Therefore, for each particle, accordingly to eq(I.20)

$$(F_\omega)_A = (F_\omega)_B = (F_\omega)_{O_1} = F_g \quad (\text{I.21})$$

Thus, every point in the revolving motion is subject to the equal and parallel centrifugal forces. Stability of the earth-moon system will require that the sum of all centrifugal and

attraction forces should be zero. While this statement is true for the centers of the earth and moon the balance does not occur in every point leading to the forces generating tides.

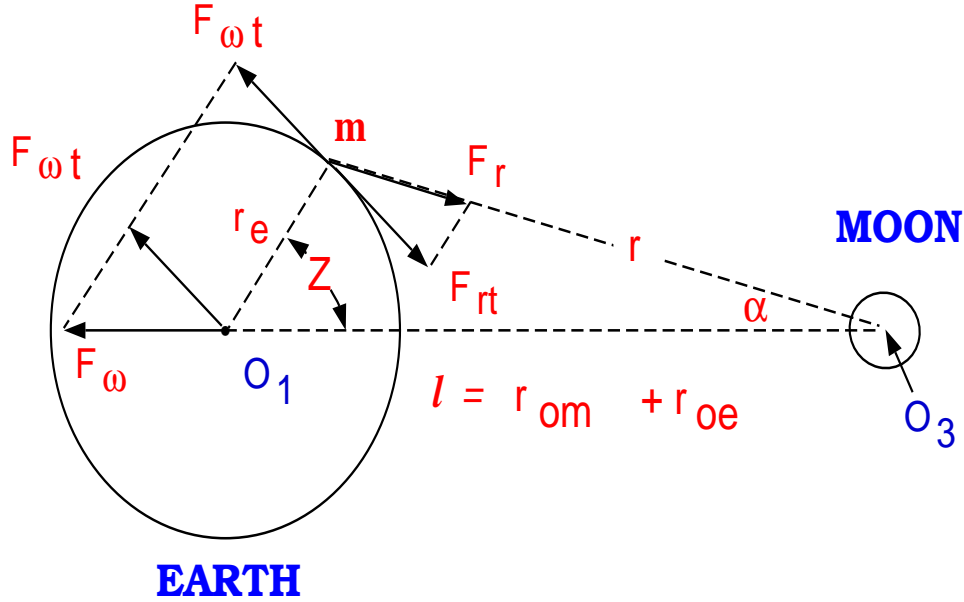


Figure I.5

Earth-moon interaction. Forces acting on the mass m located on the surface of the earth.

To find these forces let us consider a mass m located on the earth's surface (Fig. I.5). The centrifugal force F_ω is the same for the every point on the earth and according to eq(I.21) is equal to the force of attraction which moon exerts on the mass (m) located at the center of the earth;

$$F_\omega = F_g = G \frac{M_m m}{l^2} \quad (\text{I.22})$$

The force of the moon attraction acting on the mass m located on the earth's surface (see Fig.I.5) is

$$F_r = G \frac{M_m m}{r^2} \quad (\text{I.23})$$

Projection of these forces on the tangential direction to the earth's surface yield,

$$F_{\omega t} = G \frac{M_m m}{l^2} \sin Z \quad (\text{I.24})$$

and

$$F_{rt} = G \frac{M_m m}{r^2} \sin(Z + \alpha) \quad (\text{I.25})$$

Where Z is zenith angle.

Summing up these forces we arrive at the tide generating force F_t

$$F_t = GM_m m \left[\frac{\sin(Z + \alpha)}{r^2} - \frac{\sin Z}{l^2} \right] \quad (\text{I.26})$$

From the triangle O_1, m, O_3 we can find:

$$\frac{\sin Z}{r} = \frac{\sin(\pi - Z - \alpha)}{l} = \frac{\sin(Z + \alpha)}{l} \quad (\text{I.27})$$

Therefore in eq(I.26) $\sin(Z + \alpha)$ can be expressed by $\sin Z$ and

$$F_t = GM_m m \left[\frac{\sin(Z + \alpha)}{r^2} - \frac{\sin Z}{l^2} \right] = GM_m m \sin Z \left(\frac{l}{r^3} - \frac{1}{l^2} \right) \quad (\text{I.28})$$

Again using the triangle O_1, m, O_3 the distance r is defined as:

$$r^2 = l^2 + r_e^2 - 2r_e l \cos Z \quad (\text{I.29})$$

Since the **equatorial parallax ratio** for the moon r_e/l is very small number (1/60.3) and for the sun this number is even smaller

$$\frac{r_e}{l_m} = 0.01658 \quad \frac{r_e}{l_s} = 4.2615 \times 10^{-5}$$

the terms of the higher order ($r_e^2/l^2 \simeq 1/3600$) will be neglected. Developing above equation into binomial series we arrive at,

$$\frac{1}{r^3} \simeq \frac{1}{l^3} \left(1 + \frac{3r_e}{l} \cos Z \right) \quad (\text{I.30})$$

Introducing this result into the **horizontal component of the tide generating force**, yields,

$$F_t = GM_m m \sin Z \left(\frac{l}{r^3} - \frac{1}{l^2} \right) = 3GM_m m \frac{r_e}{l^3} \sin Z \cos Z = \frac{3}{2} GM_m m \frac{r_e}{l^3} \sin 2Z \quad (\text{I.31})$$

Repeating similar considerations for the forces directed along the normal direction to the surface of the earth, we arrive at,

$$F_n = F_{rn} - F_{\omega n} = 3GM_m m \frac{r_e}{l^3} (\cos^2 Z - \frac{1}{3}) \quad (\text{I.32})$$

To the tidal forces the notion of the potential (Ω) can be ascribed assuming that force per unit mass (F/m) and potential are connected as follows

$$\frac{\vec{F}}{m} = -\nabla \Omega_T \quad (\text{I.33})$$

In the system of coordinate from Fig. I.5, normal direction is away from the center of the earth and tangential direction is along $F_{\omega t}$ (compare also Fig I.2),

$$-F_t/m = -\frac{\partial\Omega_T}{\partial s} = -\frac{1}{r_e} \frac{\partial\Omega_T}{\partial Z} \quad (\text{I.34})$$

$$F_n/m = -\frac{\partial\Omega_T}{\partial n} = -\frac{\partial\Omega_T}{\partial r_e} \quad (\text{I.35})$$

Using eqs(I.31) and (I.32) the **potential of the tidal force** follows,

$$\Omega_T = \frac{3}{2}GM_m \frac{r_e^2}{l^3} \left(\frac{1}{3} - \cos^2 Z \right) \quad (\text{I.36})$$

This expression is of the first approximation. It works well for the sun-earth interaction but because of the moon proximity to the earth sometimes higher order effects become important. For such case the distance in eq(I.29) can be developed to derive next term in series;

$$\Omega_T = \frac{3}{2l}GM_m \left[\frac{r_e^2}{l^2} \left(\frac{1}{3} - \cos^2 Z \right) + \frac{r_e^3}{l^3} (3 - 5 \cos^2 Z) \right] \quad (\text{I.37})$$

One possible approach for the evaluation of **the magnitude of the tide generating force** is to compare it with the gravity force acting on mass m located on the earth's surface. Expressing gravitational constant from eq(I.4) and taking $\sin 2Z = 1$, the magnitude of F_t equals

$$F_t = mg \frac{3}{2} \frac{M_m}{M_e} \left(\frac{r_e}{l} \right)^3 \quad (\text{I.38})$$

for the moon $F_t = mg 8.43 \times 10^{-8}$ and for the sun $F_t = mg 3.87 \times 10^{-8}$

Thus the moon and the sun tide producing forces are very small in comparison to the gravity force mg . The tide generating force due to the moon is approximately 2.2 times greater than the tide generating force due to the sun. Such force will cause a man weighing 100 kg to loose (maximum) 12.3×10^{-8} of 100 kg, which is equal 12.3×10^{-3} g. To understand why such a small force has such a strong influence on the ocean dynamics it is enough to recall that the pressure force due to sea level change in the storm surge phenomena possesses the same order of magnitude.

3. EQUILIBRIUM TIDES

Equilibrium or static theory of tides has its root in the Newton works. Later it was developed by Bernoulli, Euler and MacLaurin (see **Cartwright, 1999**). Considerations involve assumption that the entire globe is covered by water, but the main tenet of the theory assumes an equilibrium between hydrostatic pressure and external (disturbing) forces. Henceforth the sea level change is defined by a simple equation;

$$-mg \frac{d\zeta^e}{ds} = F_t \quad (\text{I.39})$$

Here ζ^e denotes elevation of the free surface above mean (undisturbed) sea level. According to Fig.I.5 $ds = r_e dZ$. Introducing this dependence and expression for the F_t from eq(I.31) into eq(I.39), and using eq(I.3), we arrive at

$$-\frac{mg}{r_e} \frac{d\zeta^e}{dZ} = mg \frac{3}{2} \frac{M_m}{M_e} \left(\frac{r_e}{l}\right)^3 \sin 2Z \quad (\text{I.40})$$

and after integration

$$\zeta^e = \frac{r_e M_m}{M_e} \left(\frac{r_e}{l}\right)^3 \left(\frac{3}{2} \cos^2 Z + c\right) \quad (\text{I.41})$$

Assuming that the tide does not change the water volume, the above constant can be defined from (**Proudman, 1953**),

$$\int_0^\pi \left(\frac{3}{2} \cos^2 Z + c\right) \sin Z dZ = 0$$

Thus $c = -1/2$, and eq(I.41) defines the sea level change

$$\zeta^e = \frac{3}{2} \frac{r_e M_m}{M_e} \left(\frac{r_e}{l}\right)^3 \left(\cos^2 Z - \frac{1}{3}\right) = K \left(\cos^2 Z - \frac{1}{3}\right) \quad (\text{I.42})$$

where

$$K = \frac{3}{2} \frac{r_e M_m}{M_e} \left(\frac{r_e}{l}\right)^3$$

Using the lunar values, $K \approx 0.54$; using the solar analogs we find $K \approx 0.24$. For the combined lunar and solar effect, $K \approx 0.79$. Here we may also relate the sea level to the previously introduced (eq.I.36) tidal potential. Comparing eq.(I.36) and eq.(I.42) we arrive at the expression,

$$\Omega_T = -g\zeta^e \quad (\text{I.43})$$

According to eq.(I.42) the free surface of the ocean is deformed from the globe to an ellipsoid (see Fig.I.6). Sectors in syzygy with the moon (around points 1 and 2), where $Z = 0$ and π will depict the maximum tide, equal to:

$$\zeta^e_{max} = \frac{r_e M_m}{M_e} \left(\frac{r_e}{l}\right)^3 = \frac{2}{3} K \quad (\text{I.44})$$

Sectors in quadrature with the moon (proximity of points 3 and 4), where $Z = \pi/2$ and $3\pi/2$ will depict the minimum tide, equal to:

$$\zeta^e_{min} = -\frac{r_e M_m}{2M_e} \left(\frac{r_e}{l}\right)^3 = -\frac{1}{3} K \quad (\text{I.45})$$

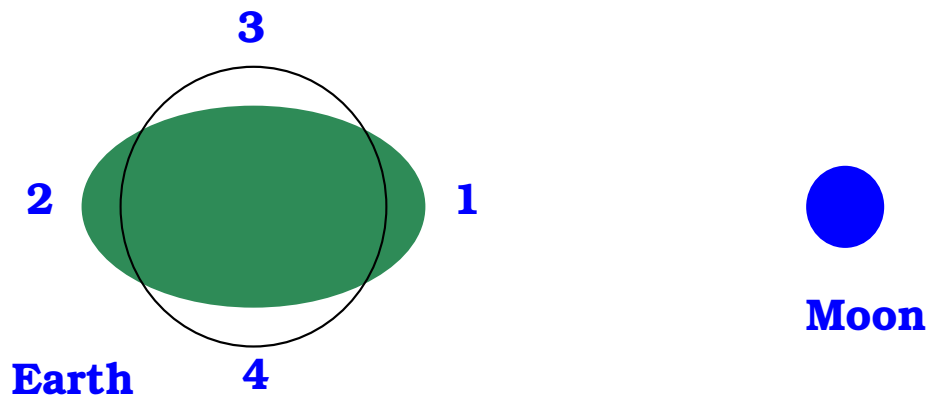


Figure I.6

Change of the shape of the oceanic free surface from the globe to an ellipsoid due to tidal forces.

Above theory has its roots in Newton research but it was first proposed by Bernoulli in 1740 who used it to explain many tidal features such as periodicity or inequalities between successive high waters and low waters. It was used by subsequent researchers and practitioners as a simple tool to recognize the basic tidal processes and for computing tide tables. This is called equilibrium or static theory. We shall investigate the dynamic tidal theory in Chapter 2. For now let's consider a channel along the earth's equator and ask a question whether the tide generated by Moon or Sun in this channel will follow the Moon or Sun without delay? The equatorial distance is close to 40,000km. If this distance is traveled in 24h the speed is 1,667km/h. The long wave whose celerity is defined as \sqrt{gH} will travel the same distance in 24h assuming the ocean depth is 21.8km. Such depth does not exist in the oceans and therefore the tidal wave will follow the Moon or Sun, but with a time delay which cannot be described solely from the static tidal theory. Actually the mean ocean depth is close to 4 km, therefore the mean celerity is approximately 200 m/s which for the wave period of 24 hr results in the wave length of 18000 km. This simple example demonstrates that the tide wavelength is much bigger than the ocean depth. For such long waves the motion will be influenced by the bottom drag thus slowing the progression of the wave forced by the moon (or sun) attraction.

The tide-producing forces and tidal potential are functions of the zenith angle. Now we turn our attention to express this angle through the system of coordinate which takes into account the position of the external body and position of the observer. In Fig.I.7 the observer at the point O, can define the point on **the celestial sphere** which is located directly above his head.

This is zenith (ZN) and an associated point in opposite direction is called nadir (ND). The line connecting the center of the earth with the north pole if continued to the celestial sphere define a point called celestial north pole (NP).

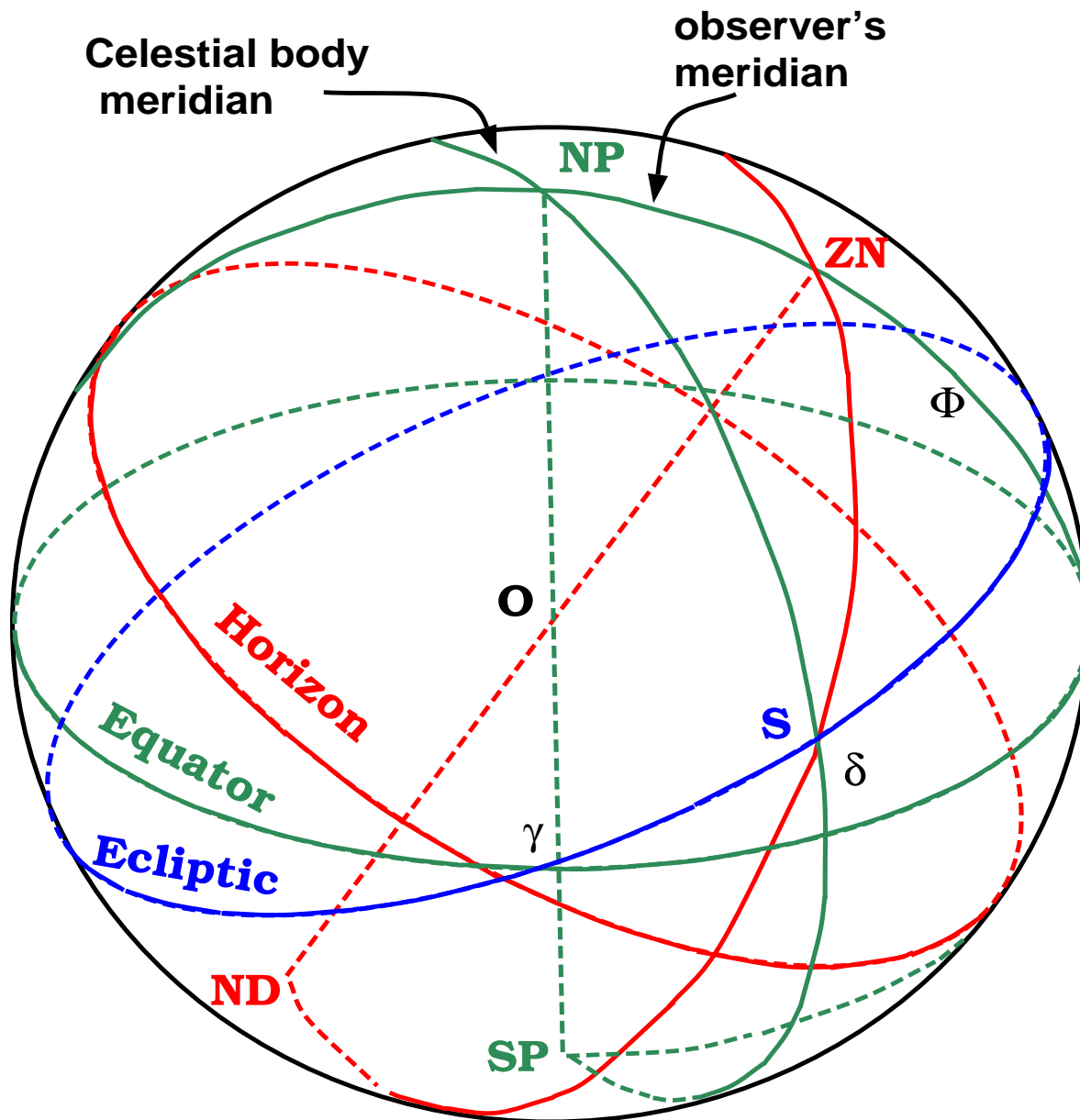


Figure I. 7

Celestial sphere. Observer is at the point O. The red color denote local system of coordinates: horizon, zenith (ZN) and nadir (ND). The green color denotes geographical coordinates: equator, north pole(NP), south pole (SP), observer's meridian (points NP, ZN, SP), Φ latitude of the observer, δ latitude of the celestial body (S), also called declination. Point γ where ecliptic crosses equator is called vernal equinox

Opposite point is called celestial south pole (SP) and continuation of the equatorial plane onto celestial sphere defines celestial equator. Therefore, the two systems of coordinates can be used on the celestial sphere: the local one related to the observer (red color in Fig.I.7), it is defined by the zenith, nadir and the plane of horizon, and the general system

(green color in Fig.I.7), defined by the north and south poles and the equatorial plane. Both system of coordinate play important role in tidal calculations. Observer's meridians joins the north pole, zenith point and south pole.

Sun, moon, planets rise in the east, climb the celestial sphere until they transit observer's meridian and then a decrease in altitude follows. The **declination** of the moon, sun or any celestial body denoted δ is the angle defined from the equator plane along the meridian of the body to the body's position on the celestial sphere. In case of the sun the plane in which it travels is called **ecliptic**. Hence for the sun the declination is an angle between the planes of the equator and ecliptic. The declination of the body is measured in the similar fashion as the latitude of the observer. To relate zenith angle Z to position of the celestial body and to position the observer let consider Fig.I.8 with the three main meridians: observer's meridian, celestial body (S) meridian (hour circle) and Greenwich meridian. First we define **the hour angle** as an angular distance along celestial equator from the meridian of an observer to the meridian of the celestial body. The hour angle is also contained in the spherical triangle S, NP, ZN. The hour angle is measured from the moment a celestial body will transit observer's meridian. It increases by 360° in one day. Moreover this angle is closely related to the longitude and to the time. The longitude of the observer's meridian is measured from the Greenwich meridian to the east. The time is measured westwards from the Greenwich meridian to the celestial body meridian (see Fig.I.8). Denoting hour angle as α , longitude of the observer as λ and time angle of celestial body as t , one can write

$$\alpha = \lambda + \frac{360^\circ}{T}t \quad (\text{I.46})$$

Here T denotes period required by celestial body to return to the observer's meridian. This time should be close to one day, but it is different for the moon and the sun. Before proceeding to develop formulas for the zenith angle let us notice an important point for the reckoning on the celestial sphere. This is a point where ecliptic intersects the equatorial surface (γ), see Fig.I.7. This the first point of Aries or vernal equinox when the sun crosses equatorial plane ascending from southern to northern hemisphere. This point is often taken as beginning of coordinate system to calculate the longitude of the celestial bodies. The longitude is calculated positive eastward from γ .

From the spherical triangle S, NP, ZN in Fig.I.7 and Fig.I.8 the zenith angle Z is defined as,

$$\cos Z = \cos \delta \cos \phi \cos \alpha + \sin \delta \sin \phi \quad (\text{I.47})$$

Defining the zenith angle in terms of the declination (of the sun or moon), the latitude of the observer, and the hour angle is a key step in placing the equilibrium sea level in terms of readily available parameters.

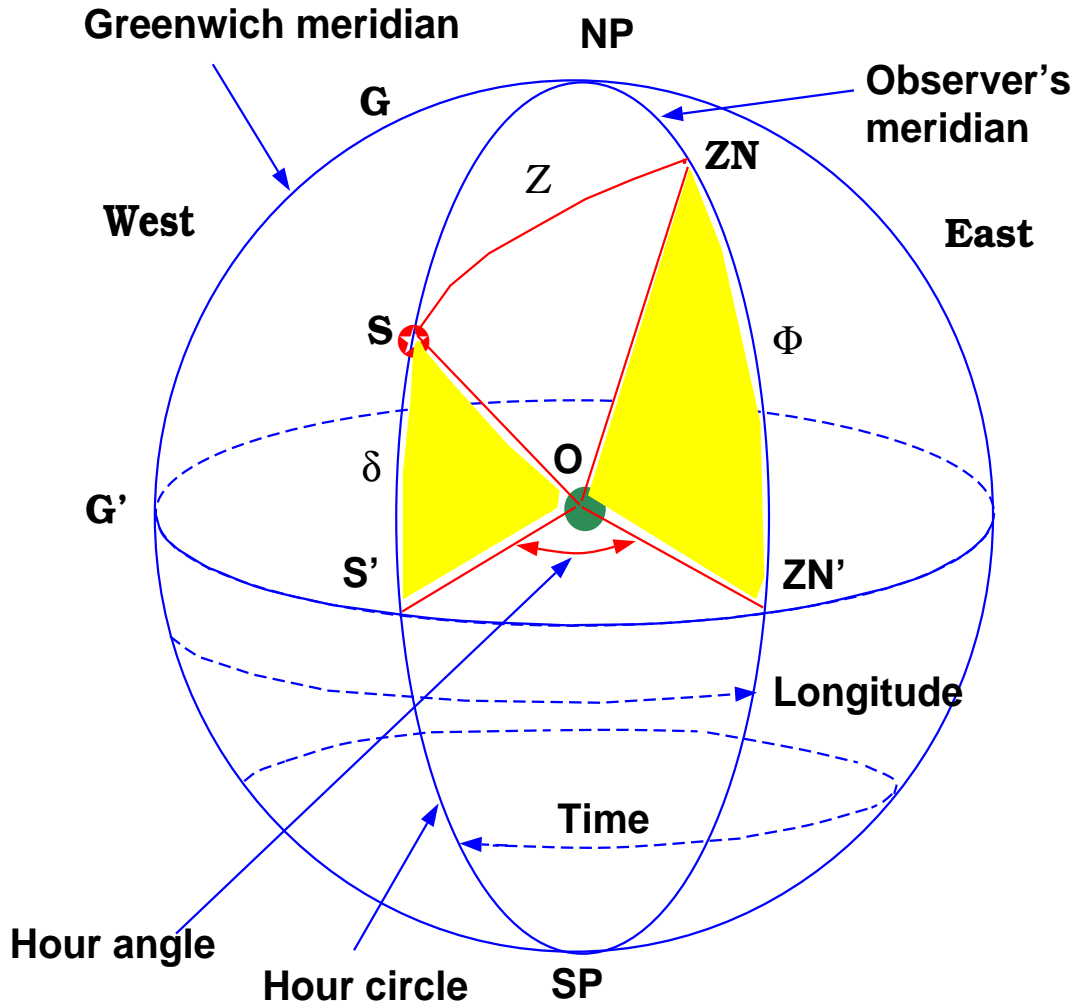


Figure I.8

Celestial sphere. Points are: O observer, ZN observer's nadir, S celestial body. Points NP, ZN, ZN' and SP depict the meridian of the observer. Points NP, S, S', SP denote the meridian of the celestial body (also called hour circle). Points NP, G, G', SP describe the Greenwich meridian. Angle ZN, O, S is zenith angle (Z). Angle ZN', O, S' is the hour angle (α), equal Time + Longitude. Longitude of the observer (λ) is measured eastward from the Greenwich meridian, time (T) of the celestial body is longitude measured from the Greenwich meridian westward to the meridian of the celestial body. Hour angle denotes time (or angle of longitude) measured westward from the observer's meridian to the meridian of the celestial body.

Using Eq. I.47, we can express I.42 (for equilibrium sea level) in terms of the angles we have just defined, which are more convenient than the zenith angle Z. Again, here we use the lunar values, but the solar equivalents could equally well have been used. Substituting

Eq. I.47 into Eq I.42, we obtain

$$\zeta^e = K(\cos^2 Z - \frac{1}{3})$$

$$= \frac{2K}{3} \left[\left(\frac{3}{2} \cos^2 \phi - 1 \right) \left(\frac{3}{2} \cos^2 \delta - 1 \right) + \frac{3}{4} \sin 2\phi \sin 2\delta \cos \alpha + \frac{3}{4} \cos^2 \phi \cos^2 \delta \cos 2\alpha \right] \quad (\text{I.48})$$

The equilibrium sea level can be split into three terms based on frequency.

The first term:

$$\zeta_{long-period}^e = \frac{2K}{3} \left(\frac{3}{2} \cos^2 \phi - 1 \right) \left(\frac{3}{2} \cos^2 \delta - 1 \right), \quad (\text{I.49})$$

does not include the hour angle (hence has no longitudinal dependence), but does depend on the declination of the celestial body (moon or sun) and the latitude of the observer. The latitude dependence produces a maximum at the poles and a minimum at the latitudes $\pm 35^\circ 16'$. The declination δ undergoes monthly variations in case of the moon, and yearly variations in case of the sun. Since $\cos^2 \delta = 0.5(1 + \cos 2\delta)$, the periodicity is one-half of the above periods. Thus this term is the source for **the semi-monthly and semi-annual periods**.

The second term:

$$\zeta_{diurnal}^e = \frac{K}{2} \sin 2\phi \sin 2\delta \cos \alpha, \quad (\text{I.50})$$

depends on the hour angle, declination and latitude. The $\cos \alpha$ dependence on the hour angle shows that this term generates **the diurnal oscillations**. The latitude dependence generates the maximum tide at the latitudes $\pm 45^\circ$ and the minimum at the equator and at the poles. The declination imparts a slow amplitude modulation and the sign of this modulation will change from positive to negative when the celestial body crosses the equator from the northern to the southern hemisphere. In all three terms the distance l changes over one month for the moon (one year for the sun) in accordance with the usual elliptical motion of celestial bodies.

The third term:

$$\zeta_{semi-diurnal}^e = \frac{K}{2} \cos^2 \phi \cos^2 \delta \cos 2\alpha, \quad (\text{I.51})$$

generates **the semi-diurnal periods** since it is proportional to $\cos 2\alpha$, and since α , the hour angle, moves through 360° each lunar (or solar in the case of the sun) day. The sea level also depends on the declination δ of the celestial body, and the latitude (ϕ) of the observer. The change of the declination over the semidiurnal period is slow. Since the semi-diurnal term is proportional to $\cos^2 \delta$ the maximum occurs when the celestial body is above the equator and is at a minimum at the poles.

Had additional terms in the approximation (Eq. I.30) been retained, higher frequency constituents (ter-diurnal etc.) would have been obtained. We may now proceed to express the equilibrium tide in terms of harmonic constituents; before doing so, however, we will review the basic elements of the earth-moon-sun orbital motions.

4. THE ORBITS OF EARTH AND MOON

The sea level produced by the tidal forcing is a function of the various astronomical parameters. As we have seen, primary among these are the revolution of the moon around the earth, the earth around the sun, and the rotation of the earth around own axis. These result in the three main tidal periods, i.e, monthly, yearly and daily.

The Earth orbit around the sun is ellipse with eccentricity $e=0.01674$. The shortest distance along the major axis at perihelion is reached on January 2, and the longest distance at aphelion is reached on July 2, see Fig. 1.9. Therefore, dependence of the equilibrium tides on the distance suggests that tide producing force due to the sun is larger in the winter. The fundamental period is equal 365.24 days. During one year the moon completes about twelve elliptical orbits around the earth with the period of 29.5 days.

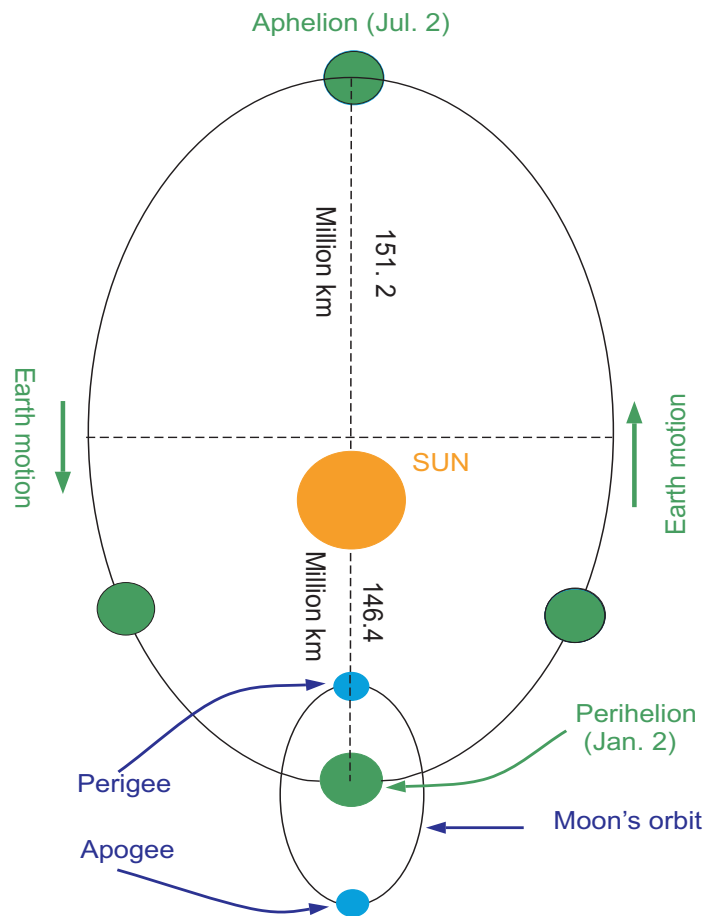


FIGURE I. 9

Sun, earth and moon in the orbital motion. Aphelion denotes the earth's longest distance from the sun, perihelion is the shortest distance. The moons shortest distance from the earth is called perigee, apogee is the longest distance

The shortest distance of the moon from the earth being called **perigee** and the longest **apogee**. The eccentricity of the moon orbit is $e=0.055$. Therefore, the moon ellipse is much more elongated than the sun orbit. This will result in the large difference of the tide producing forces between apogee and perigee. The ratio of the tidal forces at apogee and perigee is 1.4:1 for the moon and 1.11:1 for the sun. The rotation of the earth

with the period of 24 h is also one of the fundamental factors in the tide producing forces.

None of the described motions take place in the equatorial plane. Among many factors which influence tidal cycles the inclination of the ecliptic to the equator and the inclination of the lunar orbit to the ecliptic are the most important. In Fig. I.7 we have defined the declination as the inclination of the ecliptic (i.e. the sun trajectory on the celestial sphere) to the equatorial plane. The declination (or latitude) of the ecliptic changes from $23^{\circ}27'$ north of the equator to $23^{\circ}27'$ south of the equator. The moon is also slightly inclined to the ecliptic with an angle of $5^{\circ}7'$, see Fig.I. 10. The moon maximum declination changes from $28^{\circ}36'$ to $18^{\circ}18'$. These limits are reached every 18.6 years. The plane of the moon orbit intersect during one month the ecliptic plane at least at two points, called nodal points. If entire system were the earth and moon only, the moon will be always repeating the same trajectory with the same nodal points.

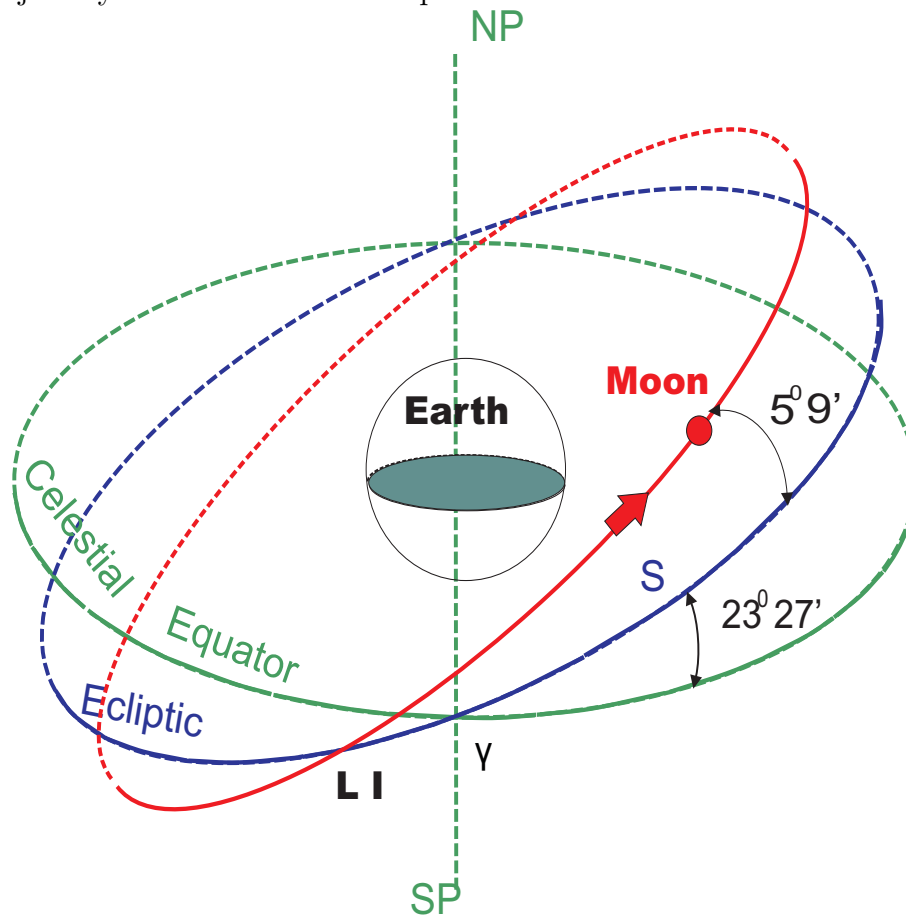


FIGURE I. 10

Moon in the orbital motion as seen from earth. Lunar orbit intersection with ecliptic (point LI) is called nodal point.

The effect of the sun is perturbation of the moon orbit. As a result of the perturbation the nodal points do not remain in the fixed position. They moved westward on the ecliptic completing one cycle in 18.6 years. This is one of the major period of the tide producing forces.

The monthly cycle of the moon latitudinal movement from $28^{\circ}36'$ N to $28^{\circ}36'$ S

should have a strong modulating effect over semidiurnal and diurnal tides with a monthly period. During one day period the moon position on the celestial sphere will change only slightly thus in the first approximation one can assume that the shape of tidal envelope is permanent over diurnal cycle. The tidal ellipsoid stays aligned with the moon, but due to the moon declination the observer in the fixed point on the rotating earth will encounter various sea level of the tidal ellipsoid, as he is carried by earth's rotation through the various phases of the tidal envelope. To describe this effect let consider the tidal ellipsoid generated by the moon (or sun) when the moon is not located in the equatorial plane so the declination is not equal to zero. We use in Fig.I.11 examples suggested by [Dronkers \(1964\)](#).

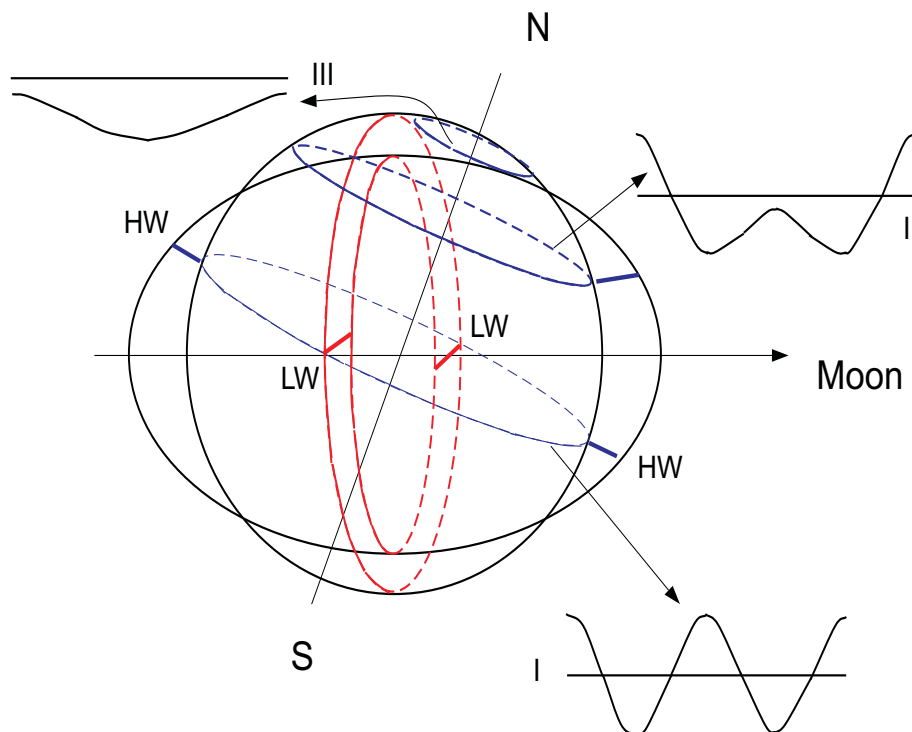


FIGURE I. 11

Tidal envelope and generation of diurnal and semidiurnal tides, according to Dronkers (1964)

Consider observational point on the earth's surface located at the equator. Due to the daily rotation this point will travel along latitudinal circle (equator). The high water level (HW) occurs when the observational point pass under the moon (moon passes the observer's meridian) and when this point is on the opposite side of the earth. The low water level (LW) occurs when the moon rises or sets in the observational point. Henceforth, the observational point will encounter two low and two high sea levels during one day, see tidal record I in Fig.I.11. At the equator the semidiurnal tide is quite symmetrical. This symmetry is broken at higher latitudes. Record II in Fig.I.11 depicts quite strong daily inequality in the semidiurnal tides. Away from the equator the diurnal tides are growing stronger and eventually at the high latitudes (record III in Fig.I.11) this period completely dominates the tidal record. Daily inequalities can be studied precisely by using formulas for the semidiurnal equilibrium tide (eq I.50) and for the diurnal tide (eq I.51).

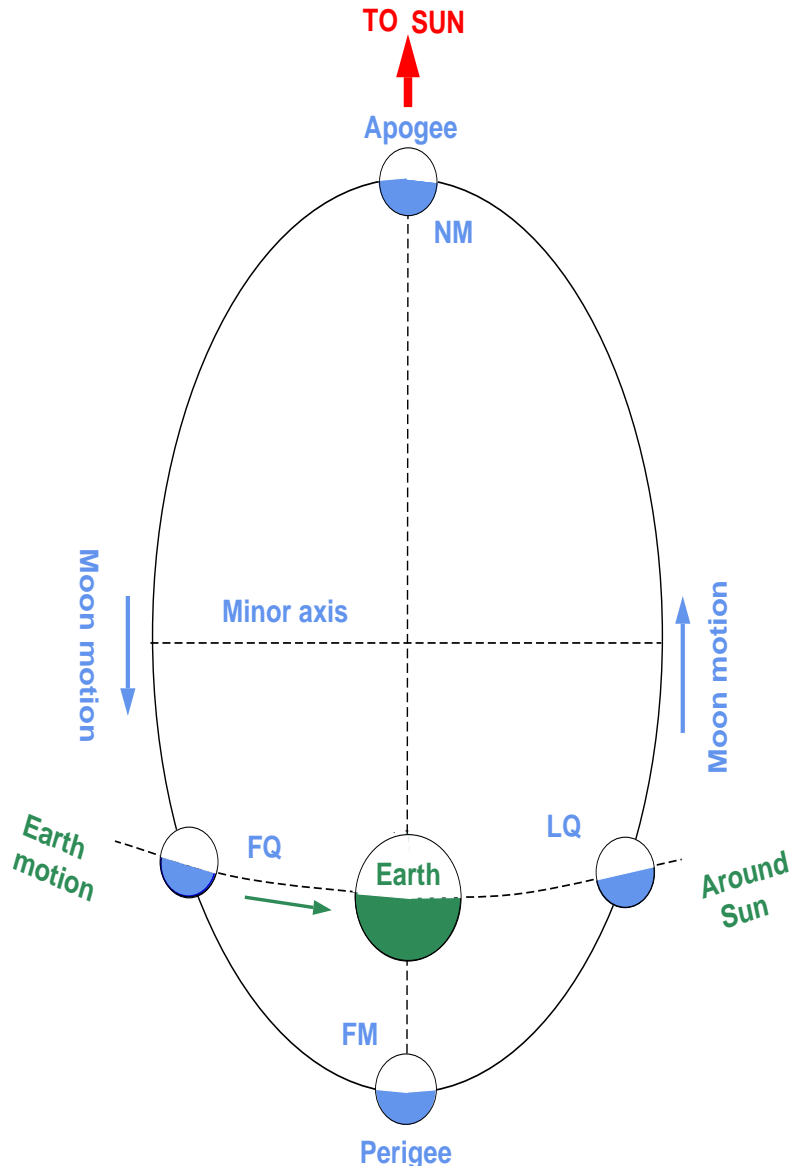


FIGURE I. 12a

Moon orbital motion. Maximum tides (spring tide) occur at both new (NM) and full moon (FM). Minimum tides (neap tides) occur at the first quarter (FQ) and the last quarter (LQ). The alignment of the sun, earth and moon is called syzygy.

The moon revolves around the earth in one synodic cycle from the new moon (NM) through the first quarter (FQ), the full moon (FM), the last quarter (LQ) and back to the new moon in one synodic cycle equal to 29.53 days (Fig.I.12a). The various positions of these celestial bodies will result in the various magnitude of the tide producing force. The maximum of tides (spring tide), occur at the new moon and at the full moon when the earth moon and sun are in one line. This alignment is called syzygy. Minimum of tide (neap tides) occur at the first and last moon quarters.

14.75 days.

The diurnal period related to the moon producing forces is equal about 24.84 h. Every day transit of the moon across the observer's meridian is 0.84 h (50 min) later. To deduce the moon successive transits let's consider closely the motion of the earth and moon during one month - Fig.I.13. The moon moves counterclockwise about 12.5° per day, because the earth's rotation takes place in the same direction, the observer's meridian (see Fig.I.13) will not be aligned with the moon in 24 h period. Additional, about 12.5° will require about 50 min, since one degree (on the earth) corresponds to 4 min time lag. Moon periodic motion around the earth defined by **synodic month** (period between two identical moon phases) is not equal to the lunar sidereal month (sidereal meaning with respect to a fixed reference frame). The reason for this difference is that the earth and moon jointly revolve around the sun. The results of combining the moon monthly revolution and the earth's annual revolution are shown in Fig.I.13, according to **Macmillan (1966)**. During one sidereal month the earth moves from point A to B, thus advancing about 29° . After that period the star, the moon and the earth are aligned again. The period is called sidereal month and is equal 27.3 days. However the sun is not aligned, because during the one sidereal month the earth moved around the sun, for the moon to be in line with the sun, it must revolve additionally about 29° or about 2.2 days, thus completing one synodic month in about 29.5 days. This alignment will also result in the strongest tides since the shortest distance from the earth to both the sun and moon will occur during new moon (NM) perigee. Generally, we can deduce from the above considerations that the time of one complete revolution of the moon around the earth will depend on the system of reference. Frequently, together with the synodic and sidereal definitions, the nodical and anomalistic months are used as well (see below table).

Table I.1 Duration of the month according to the reference system

Reference system	Month	Duration
Moon node	Nodical	27.2122 d
Star	sidereal (tropical)	27.3216 d
Moon perigee	anomalistic	27.5546 d
Moon phase	synodic	29.5306 d

5. THE HARMONIC CONSTITUENTS OF THE EQUILIBRIUM TIDE

The sea level in eq(I.48) depends in a quite complicated fashion on the latitude of the observer, the distance to and declination of the celestial body, and the hour angle, with the latter three all being functions of time. This does not make a convenient basis for the representation of the tides. The harmonic model (Eq.I.52) proposed by **Thomson (Lord Kelvin) (1882)**, and developed chiefly by **Darwin (1901)** and **Doodson (1921; reprinted in 1954)** enabled tidal scientists to analyse tidal records in terms of a set of parameters (such as the current longitude of the moon) which vary in a simple clocklike fashion. Doodson's (1921) tidal potential model became the basis for most tidal analysis

and prediction schemes through the 20th century. Summaries may be found in [Neumann and Pierson \(1966\)](#), [Dronkers \(1964\)](#), [Dietrich et al. \(1980\)](#), and [Pugh \(1987\)](#). The theory was put on a more modern foundation by [Cartwright and Tayler \(1971\)](#), with minor corrections published by [Cartwright and Edden 1973](#) (collectively "CTE"). [Kantha and Clayson \(2000\)](#) give an excellent elaboration of the methods of CTE.

The names of the basic tidal constituents (M_2 , K_1 , etc.) originated with Sir William Thomson (Lord Kelvin) and Sir George Darwin in the 19th and early 20th centuries. [Cartwright \(1999\)](#), pp. 100-103, provides an interesting account of how the convention developed. Aside from "S" for solar and "M" for moon (lunar), the rationale for the letters is not obvious. The subscript specifies the species. The harmonic analysis introduced by Thomson found primary application in harmonic development of the tide-generating potential by [Darwin \(1901\)](#) who introduced the basic tidal constituents and their names. Further extension to about 400 constituents was made by [Doodson \(1921\)](#). In Tables I.3 and I.4 we define 11 major constituents in the semidiurnal, diurnal and long period range of oscillations. General symbols like M and S refer to lunar and solar origin. The subscripts 1 and 2 refer to diurnal and semidiurnal species respectively. Hence **M_2 is semidiurnal constituent due to the moon attraction** with the period equal to half of the mean lunar day. **S_2 is semidiurnal constituent due to sun attraction** with the period equal to half of the solar day. The constituent **N_2 is called the lunar elliptic semidiurnal** because it depends on the changes of distance. The constituents **K_2 is called a luni-solar declinational semidiurnal constituent** because it depend on the changes in the moon and sun declination and not on the changes in distance. Both declinational constituent due to the sun and the moon have the same period. Similarly **K_1 constituents** for the moon and sun have the same period, therefore they are combined into one luni-solar diurnal constituent. It is interesting to see from the Tables I.3 and I.4 that the main solar and lunar constituents are not identified as S_1 and M_1 . These should have periods of the mean solar day and the mean lunar day but these diurnal constituents depend on the declination as $\sin 2\delta$, hence the input of these terms is equal zero. Three long-period constituents M_f , M_m and S_{sa} are considered. The most important constituent is the **lunar fortnightly (Mf)** with period 13.661 day. It is associated with the variation of the moon's monthly declination. The **monthly lunar period Mm** is related to the monthly variation of the moon's distance and therefore this period is also named as elliptic. This period is equal to 27.555 day it is so called anomalistic month.

Doodson represented the equilibrium sea level as a sum of n individual harmonic waves, each having one of two forms:

$$\zeta_n^e = \frac{1}{g} C_n S_D G_{ms} F_D(\phi) \cos(V_n(t) + d_1 \lambda) \quad \text{or} \quad \zeta_n^e = \frac{1}{g} C_n S_D G_{ms} F_D(\phi) \sin(V_n(t) + d_1 \lambda). \quad (\text{I.52})$$

The meaning of the terms in Eq.I.52 are as follows.

C_n is the relative amplitude of the n^{th} wave. Values of C_n are listed in a series of "Schedules" in [Doodson \(1921\)](#). The value listed for M_2 , for example, is 0.90812.

S_D is a scaling factor chosen such that all the products $S_D G_m F(\phi)$ (which Doodson

referred to as the "geodetic coefficients") have identical maximum values. For long-period, diurnal, and semi-diurnal harmonics, the values of S are 0.5, 1, and 1 respectively, but for ter-diurnal and higher harmonics, the values range between 0.125 and 3.079.

G_{ms} is a constant. For lunar and solar terms respectively it takes on the values

$$G_m = \frac{3}{4}GM_m \frac{R_e^2}{\langle l_m \rangle^3} \quad \text{and} \quad G_s = \frac{3}{4}GM_s \frac{R_e^2}{\langle l_s \rangle^3}$$

where the angle brackets indicate mean values.

$F_D(\phi)$ is a function of earth latitude. For M_2 , $F_D(\phi) = \cos^2 \phi$.

$V_n(t)$ is the phase of the n^{th} equilibrium harmonic on the Greenwich meridian. Further discussion may be found in the following section.

$d_1\lambda$ advances the phase with west longitude λ . $d_1 = 0, 1, 2$ for long-period, diurnal, and semi-diurnal tides respectively.

The lunar tidal potential has an 18.6 year periodicity known as the "regression of the lunar nodes" (see Sec. 7) which for time series of less than 18.6 years is usually accounted for using a factor and phase shift known as "f and u". Also, in order to let the phase advance in time at a frequency ω_n , we add a term $\omega_n(t)$, where t is the time elapsed since $t_0 = 0000$ hours UT, and we explicitly reference the phase to that time by writing $V_n(t_0)$. With these changes, and assuming appropriate phase changes have been made in order to use exclusively cosine terms with positive coefficients, Eq.I.52 becomes (for all terms, lunar and solar):

$$\zeta_n^e = \frac{f_n}{g} C_n G_m F(\phi) \cos(\omega_n t + V_n(t_0) + u_n) \quad (\text{I.53})$$

The full equilibrium sea level with n harmonics, using **Doodson's (1921)** scalings, is given by the sum:

$$\zeta^e = \sum_n \frac{f_n}{g} C_n G_m F(\phi) \cos(\omega_n t + V_n(t_0) + u_n) \quad (\text{I.54})$$

The expansion of CTE results in a simpler scaling and more accurate coefficients, although for the most part the differences were found to be minor. For example, CTE lists $H_{M_2} = 0.63192$ for the coefficient based on the most recent data (centered on 1960). When scaled to compare with Doodson's coefficient ($C_{M_2} = 0.90812$), CTE found that $H_{M_2} = 0.90809$. More recently, other authors (e.g. **Tamura, 1987**) have taken the CTE expansion to higher order, adding some additional precision. With CTE coefficients, Eq.I.53 is written

$$\zeta_n^e = S_{CTE} H_n F_{CTE}(\phi) \cos(\omega_n t + V_n(t_0) + u_n). \quad (\text{I.55})$$

Here

$S_{CTE} = \left(\frac{5}{4\pi}\right)^{1/2}$, $\frac{3}{2}\left(\frac{5}{24\pi}\right)^{1/2}$, and $3\left(\frac{5}{96\pi}\right)^{1/2}$ for long-period, diurnal, and semi-diurnal harmonics respectively, and

$F_{CTE}(\phi) = \frac{3}{2}\cos^2\phi - 1$, $\sin 2\phi$, and $\cos^2\phi$ for long-period, diurnal, and semi-diurnal harmonics.

For example, M_2 then becomes:

$$\zeta_{M_2}^e = 3\left(\frac{5}{96\pi}\right)^{1/2} H_{M_2} \cos^2\phi \cos(\omega_{M_2}t + V_{M_2}(t_0) + u_{M_2}). \quad (\text{I.56})$$

The equilibrium sea level, using the scaling and coefficients of CTE, including the f and u factors, is given by the sum:

$$\zeta^e = \sum_n f S_{CTE} H_n F_{CTE}(\phi) \cos(\omega_n t + V_n(t_0) + u_n) \quad (\text{I.57})$$

The period of each term in the harmonic model derives directly from the orbital and rotational periods of discussed in Sec. 4. These include:

the mean solar hour (msh) (lunar hours and lunar days may also be used, but see below),

the sidereal month (period of lunar declination), 27.321582 mean solar days,

the tropical year (period of solar declination), 365.24219879 mean solar days,

the period of the lunar perigee, 8.8475420 years

the period of the lunar node, 18.613188 years, and

the period of the solar perihelion, 20,940 years.

The "year" in the definitions of the latter three is the Julian year (1 Julian Year 365.24219879 days). Although the longitude of the perihelion changes by less than $.02^\circ$ per year, this is enough to cause a noticeable effect in the frequencies of certain constituents, and that is why it must be retained.

The letters s , h , p , N and p_s (sometimes p' or p_1) are well-established in the tidal literature, but usage varies. Most commonly they are time-dependent functions of longitude on the celestial sphere and have units of degrees. In other words, many authors use s for the longitude of the moon, p for the longitude of the moon's perigee, etc. Elsewhere they are understood to mean the time rate of change of those longitudes, in which case the meaning is clear only if the dot notation is used (as in " \dot{s} "), but unfortunately the dot is sometimes omitted. We use \dot{s} , \dot{h} , \dot{p} , \dot{N} , and \dot{p}_s for the rates of change, and $s(t)$, $h(t)$, $p(t)$, $N(t)$, and $p_s(t)$ to indicate the time-dependent longitudes.

It is conventional in tidal practice to use "speed", usually in degrees per msh, rather than radians per second, as the unit of frequency. In terms of speed (with msh = mean solar hour and msd = mean solar day), the above periodicities become:

$$\omega_t = 15.0^\circ/msh = 360^\circ/msd$$

$$\omega_m = 14.4920521^\circ/msh = 360^\circ/\text{mean lunar day},$$

$$\dot{s} = 5.490165 \times 10^{-1} \text{ }^\circ/msh = 13.17644^\circ/msd = 360^\circ/\text{sidereal month},$$

$$\dot{h} = 4.106863 \times 10^{-2} \text{ }^\circ/msh = 0.98565^\circ/msd = 360^\circ/\text{tropical year},$$

$\dot{p} = 4.641878 \times 10^{-3} \text{ }^\circ/msh = 0.11140^\circ/msd = 360^\circ/\text{lunar perigee cycle}$,
 $\dot{N} = 2.206413 \times 10^{-3} \text{ }^\circ/msh = 0.052955^\circ/msd = 360^\circ/\text{nodal cycle}$, and
 $\dot{p}_s = 1.96125 \times 10^{-6} \text{ }^\circ/msh = 0.000047^\circ/msd = 360^\circ/\text{solar perihelion cycle}$.

Conversion between speed in $^\circ/hour$ and circular frequency in radians per second is straightforward: e.g., for the K_1 wave:

$$\omega_{K1} = 15.041068^\circ \times (\pi/180)/3600. = 0.7292110^{-4} s^{-1} \quad (\text{I.58})$$

In the list above, two different types of day are used. Both are slightly longer than a sidereal day as a result of the orbit of the earth around the sun (in the case of the solar day) and the orbit of the moon around the earth (in the case of the lunar day). Hence the solar and lunar diurnal frequencies are related to the sidereal frequency ω_s by (respectively) $\omega_t = \omega_s - \dot{h}$ and $\omega_m = \omega_s - \dot{s}$. Table I.2 shows the respective periods.

Table I.2 Fundamental Daily Periods (Platzman (1971))

Period (msd)	Description
$360^\circ/\omega_s = 0.997270$ ($23^h 56^m 4^s$)	Sidereal day
$360^\circ/\omega_t = 1$	Mean solar day
$360^\circ/\omega_m = 1.035050$ ($24^h 50^m 28^s$)	Mean lunar day

The frequency of any tidal constituent can be computed by:

$$\omega_n = d_1\omega_t + d_2\dot{s} + d_3\dot{h} + d_4\dot{p} + d_5\dot{N} + d_6\dot{p}_s \quad (\text{I.59})$$

where the numbers ($d_1, d_2, d_3, d_4, d_5, d_6$) are a set of six small integers known as the "Doodson Number" of the constituent. The **Doodson Number** identifies both the speed and the equilibrium phase.

The Doodson Number was originally written NNN.NNN where each N was an integer in the range 0 - 9 (or "X" for ten). In some tabulations five was added to each integer (except the first) to avoid negative values. This practice is no longer of much value, so we will write the Doodson Number simply as ($d_1, d_2, d_3, d_4, d_5, d_6$). Each Doodson Number takes on a value ranging between -12 and +12 (between -4 and +4 for most major tidal constituents).

The first digit of the Doodson Number is the species of the constituent. The second and third digits (d_2 and d_3) of the Doodson Number take on different values depending on whether solar or lunar time is being used, and this may not always be spelled out. For example, the Doodson Number for M_2 would be written (2 0 0 0 0 0) or (2 2 2 0 0 0) respectively*. The first one tells us that the constituent oscillates twice per lunar day, as one expects for M_2 . The latter tells us that the same constituent oscillates twice per solar day, less twice per month, plus twice per tropical year, which of course must come to the

same thing. Given a Doodson Number of (2 0 0 0 0 0), there is no way to know if it is M_2 or S_2 , unless we are told whether the "2" refers to "twice per lunar day" or "twice per solar day". Obviously the time base must always be clearly specified. Unless otherwise specified, we will use the solar-day based Doodson Numbers exclusively. Conversion of the Doodson Number from one time base to the other is quite simple.

*In a table that adds five to all digits except the first, the representation of M_2 is either (2 5 5 5 5 5) or (2 3 7 5 5 5).

Table I.3 Doodson Numbers and speeds of the major tidal constituents

Constituent semidiurnal	Doodson number, solar time						speed in °/h
	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	
M_2	2	-2	2	0	0	0	28.984
S_2	2	0	0	0	0	0	30.000
N_2	2	-3	2	1	0	0	28.440
K_2	2	0	2	0	0	0	30.082
diurnal							in °/h
K_1	1	0	1	0	0	0	15.041
O_1	1	-2	1	0	0	0	13.943
P_1	1	0	-1	0	0	0	14.958
Q_1	1	-3	1	1	0	0	13.399
long period							in °/h
Mf	0	2	0	0	0	0	1.098
Mm	0	1	0	-1	0	0	0.544
Ssa	0	0	2	0	0	0	0.082

The following may be used to convert to solar time:

Diurnal constituents ($d_1 = 1$):

$$d_2(\text{solar}) = d_2(\text{lunar}) - 1, \text{ and } d_3(\text{solar}) = d_3(\text{lunar}) + 1,$$

Semi-diurnal constituents ($d_1 = 2$):

$$d_2(\text{solar}) = d_2(\text{lunar}) - 2, \text{ and } d_3(\text{solar}) = d_3(\text{lunar}) + 2,$$

Ter-diurnal constituents ($d_1 = 3$):

$$d_2(\text{solar}) = d_2(\text{lunar}) - 3, \text{ and } d_3(\text{solar}) = d_3(\text{lunar}) + 3,$$

...and so forth for higher frequency constituents. Note that only d_2 and d_3 differ.

For example, to compute the speed of S_2 , with solar day Doodson Number (2 0 0 0 0 0), and lunar day Doodson Number (2 2 -2 0 0 0), we have:

$$\omega_{S_2} = 2\omega_t = 30^\circ/\text{msh}, \text{ or}$$

$$\omega_{S_2} = 2\omega_m + 2\dot{s} - 2\dot{h} = 30^\circ/\text{msh}.$$

For M_2 and K_1 , using only the solar day Doodson Numbers, $\omega_{M_2} = 2\omega_t - 2\dot{s} + 2\dot{h} = 28.984^\circ/\text{h}$, $\omega_{K_1} = \omega_t + \dot{h} = 15.041068^\circ/\text{h}$

In case of nonlinear interactions through the advective terms or bottom friction terms in the equation of motion the new constituents are generated. These are called overtides or compound tides. Compound tides have speeds that are linear combinations of the basic

constituents. For example, $2SK_2$ has twice the speed of S_2 less that of K_2 , i.e. $2\omega_{S_2} - \omega_{K_2}$ (some authors write this as " $2S_2 - K_2$ "). Using the Doodson Numbers for S_2 and K_2 , (2 0 0 0 0 0) and (2 0 2 0 0 0) respectively, ω_{S_2} and ω_{K_2} are expanded as:

$$\omega_{S_2} = 2\omega_t, \text{ and}$$

$$\omega_{K_2} = 2\omega_t + 2\dot{h}. \text{ Then the expression is written}$$

$$2\omega_{S_2} - \omega_{K_2} = 2(2\omega_t) - 2\omega_t - 2\dot{h} = 2\omega_t - 2\dot{h},$$

indicating a constituent with Doodson Number (2 0 -2 0 0 0), i.e. $2SK_2$. Had lunar day Doodson Numbers been used, the result would have been (2 2 -4 0 0 0), which is of course also $2SK_2$. The names of the compound constituents, described in [Rossiter and Lennon \(1968\)](#) includes the more basic constituents from which they arise. A few examples shows the logic behind them. While the names are suggestive of the origins, they don't always identify them completely.

$2MS_6$, which oscillates six times per day, arises from M_2 and S_2 . The speed is determined as $2\omega_{M_2} + \omega_{S_2}$, that is, twice the speed of M_2 plus the speed of S_2 .

$2MQ_3$ arises from M_2 and Q_1 and the speed is given by $2\omega_{M_2} + \omega_{Q_1}$. $2MQ_3$ is a bit unusual because most compound tides involve only constituents of the same species (i.e., all diurnal or all semi-diurnal).

$2(MN)S_6$ arises from M_2 , N_2 , and S_2 and the speed is given by $2\omega_{M_2} + 2\omega_{N_2} - \omega_{S_2}$. Note that the sign can be negative, and that the parenthesis indicates that M_2 and N_2 have the same sign. The only way for the speeds to add up to six is to have factors of positive two on M_2 and N_2 , and a factor of negative one on S_2 . Thus, $2 \times 2 + 2 \times 2 - 2 = 6$.

$V_n(t_0)$, the phase of the equilibrium tide at a given place and time, can be written as a function of $s(t)$, $h(t)$, $p(t)$, $N(t)$, $p_s(t)$:

$$V_n(t_0) = d_2s(t_0) + d_3h(t_0) + d_4p(t_0) + d_5N(t_0) + d_6p'(t_0) + \Phi_n \quad (\text{I.60})$$

In the expansion, the angle Φ arises due to the requirement that the summation (Eq. I.52) is over cosines only (not sines) (adding Φ , which is always a multiple of $\pi/2$, converts a sine term to a cosine), and also that the tabulated coefficients be positive. The latter is achieved by applying one of the following identities: $\cos(\theta - 90^\circ) = \sin \theta$, and $\cos \theta = -\cos(\theta - 180^\circ)$.

In Eq.I.60 we have written t_0 rather than t to emphasise that the expression is normally evaluated at t_0 , the start of a solar day (i.e., at 0000 Hours UT). At that time, α , the hour angle of the sun, is zero. If $V_n(t)$ is required at some time other than 0000 Hours UT, a term $d_1\omega_n t$ could be added, where ω_n is the speed of the n^{th} constituent and t is the time in hours since the start of the day.

The values in Eq.I.60 are sometimes referred to collectively as the "astronomical argument" of constituent n . For example, for O_1 , with Doodson Number (1 -2 1 0 0 0) (note that the Doodson Numbers used in the computation of $V_n(t_0)$ must be in solar time), and $\Phi = 270^\circ$ (see Table I.4), the astronomical argument would be written $-2s+h+270$. Hence, the equilibrium phase for O_1 and M_2 are simply:

$$V_{O_1}(t_0) = -2 s(t_0) + h(t_0) + 270^\circ$$

and

$$V_{M_2}(t_0) = -2 s(t_0) + 2h(t_0) \tag{I.61}$$

Values of $s(t_0)$ and $h(t_0)$ are easily computed using one of the equations of time given at the end of the chapter. In this notation, Eq.I.56 is written:

$$\zeta_{M_2}^e = 3 \left(\frac{5}{96\pi} \right)^{1/2} H_{M_2} \cos^2 \phi \cos(\omega_{M_2} t + 2s(t_0) - 2h(t_0) + u_{M_2}).$$

Table I.4 Tidal parameters of the major tidal constituents

Constituent	Astronomical argument*	Speed °/h
Semidiurnal		
M ₂	$-2s + 2h$	28.984
S ₂	0	30.000
N ₂	$-3s + 2h + p$	28.440
K ₂	$2h$	30.082
Diurnal		
K ₁	$h + 90^\circ$	15.041
O ₁	$-2s + h - 90^\circ$	13.943
P ₁	$-h - 90^\circ$	14.958
Q ₁	$-3s + h + p - 90^\circ$	13.399
Long period		
Mf	$2s$	1.098
Mm	$s - p$	0.544
Ssa	$2h$	0.082

* Some tables add 360° to the phase, e.g. list 270° instead of -90°.

In Table I.5 we define 11 major constituents in the semidiurnal, diurnal and long period ranges. The term ζ_n^e represents the magnitude of the constituent in the tidal potential. The largest such term is M₂, with amplitude 0.24 m, followed by K₁, with amplitude 0.14 m. The frequencies of the constituents in the harmonic model (Eq. I.57) can be expressed either by speeds in °/h (Table I.4) or radian/s (Table I.5). From Table I.5 tidal constituents are clustered around the main diurnal and semidiurnal periods.

Table I.5 Parameters of the Major Tidal Constituents

Constituent		Period ¹	Freq. (s ⁻¹)	ζ_n^e (m)
Semidiurnal				
Principal Lunar	M ₂	12.421 msh	1.40519·10 ⁻⁴	0.242334
Principal Solar	S ₂	12.000 msh	1.45444·10 ⁻⁴	0.112841
Elliptical Lunar	N ₂	12.658 msh	1.37880·10 ⁻⁴	0.046398
Declination Luni-Solar	K ₂	11.967 msh	1.45842·10 ⁻⁴	0.030704
Diurnal				
Declination Luni-Solar	K ₁	23.934 msh	0.72921·10 ⁻⁴	0.141565
Principal lunar	O ₁	25.819 msh	0.67598·10 ⁻⁴	0.100574
Principal solar	P ₁	24.066 msh	0.72523·10 ⁻⁴	0.046843
Elliptical lunar	Q ₁	26.868 msh	0.64959·10 ⁻⁴	0.019256
Long-Period				
Fortnightly Lunar	M _f	13.661 msd	0.053234·10 ⁻⁴	0.041742
Monthly Lunar	M _m	27.555 msd	0.026392·10 ⁻⁴	0.022026
Semiannual Solar	S _{sa}	182.621 msd	0.0038921·10 ⁻⁴	0.019446

¹ msh: mean solar hour msd: mean solar day

6. THE ORIGINS OF MAJOR TIDAL VARIATIONS

The fortnightly or spring/neap cycle of tides may be understood in terms of two signals going in and out of phase. This phenomenon is known as a "beats" or "beating". When a pair of slightly-different frequencies of similar amplitude add together, their sum undergoes a regular cycle between near-zero magnitude (neaps), and a magnitude equal to the sum of the pair (springs). This may be seen using the identity

$$\begin{aligned} \cos(\omega_1 t) + \cos(\omega_2 t) &= 2 \cos \frac{1}{2}(\omega_1 - \omega_2)t \cos \frac{1}{2}(\omega_1 + \omega_2)t = \\ &= 2 \cos \omega_m t \cos \omega_c t \end{aligned} \quad (\text{I.62})$$

where $\omega_c = \frac{1}{2}(\omega_1 + \omega_2)$ and $\omega_m = \frac{1}{2}(\omega_1 - \omega_2)$ are carrier and modulation frequencies. The shorter (carrier) and the longer (modulation) periods are expressed as,

$$\frac{2\pi}{T_c} = 0.5\left(\frac{2\pi}{T_1} + \frac{2\pi}{T_2}\right) \quad \text{or} \quad T_c = \frac{2T_1 T_2}{T_1 + T_2} \quad (\text{I.63a})$$

$$\frac{2\pi}{T_m} = 0.5\left(\frac{2\pi}{T_1} - \frac{2\pi}{T_2}\right) \quad \text{or} \quad T_m = \frac{2T_1 T_2}{T_2 - T_1} \quad (\text{I.63b})$$

This shorter wave period T_c is modulated by the longer wave period T_m . The most important tidal beat is between M₂ and S₂, in whose case the shorter-period wave is semidiurnal. The frequencies of M₂ and S₂ are 1.9322 d⁻¹ and 2.0 d⁻¹ respectively (where d is mean solar day); hence the modulation has frequency 0.0678 d⁻¹, the inverse of which, 14.75 days, is the beat period (fortnightly cycle). Of course, a power spectrum of the two

combined signals would contain no power at the beat period. The fortnightly tide "Msf" is caused by a fortnightly variation in the declination of the moon.

If the earth, sun and moon orbits were circular rather than elliptical, and in the same plane, and the earth's axis of rotation were perpendicular to that plane, the only important tidal frequencies would be M_2 , S_2 , and their multiples and compounds (M_6 , MS_4 , etc.). All the other tidal frequencies arise because of the orbital ellipticities, the angle between the orbital planes, and the tilt of the earth's axis. All the other tidal frequencies arise from these three variations and as a consequence can be expressed as simple linear combinations of their rates of change.

The modulation of the amplitude (a_c) of the harmonic wave ($a_c \cos(\omega_c t)$) by a signal of lower amplitude and frequency ($a_m \cos(\omega_m t)$) can be represented as

$$(a_c + a_m \cos(\omega_m t)) \cos(\omega_c t)$$

which we can write

$$a_c \cos(\omega_c t) + a_m \cos(\omega_m t) \cos(\omega_c t).$$

Then using $2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$, we have

$$a_c \cos(\omega_c t) + \frac{a_m}{2} \cos(\omega_c + \omega_m)t + \frac{a_m}{2} \cos(\omega_c - \omega_m)t. \quad (\text{I.64})$$

Thus we have our original frequency (whose amplitude is unaffected), plus two "sidebands" whose frequencies are slightly above and below it. This type of modulation is associated with many of the constituents of the equilibrium tide. For example, the ellipticity of the moon's orbit adds two sidebands (N_2 and L_2) to the M_2 constituent due to the changing distance between the centers of mass of earth and moon. The period of M_2 is 12.421 hours, and the period required for the moon to complete an orbit (i.e., to reach successive perigees, known as the anomalistic month) is 27.5546 days. Thus, the M_2 speed is $28.984^\circ/\text{hour}$ and the modulation speed is $0.544^\circ/\text{hour}$. In accordance with Eq. I.64, our new sideband speeds are $28.984 \pm 0.544^\circ/\text{hour}$, i.e., 28.44 and $29.53^\circ/\text{hour}$ – the speeds associated with N_2 and L_2 . N_2 and L_2 are known as the larger and smaller elliptical lunar semidiurnal constituents. If effects of angular speed are considered, it can be shown that the tidal potential of N_2 is larger than that of L_2 .

The terms T_2 and R_2 are the solar equivalents of N_2 and L_2 . Their origin is in the ellipticity of earth's orbit around the sun and they are known respectively as the larger and smaller solar elliptic semidiurnals. Their speeds are $\omega_{S_2} + \omega_{S_a}$ and $\omega_{S_2} - \omega_{S_a}$ ($30. \pm 0.041 = 29.941$ and $30.041^\circ/\text{hour}$). T_2 and R_2 are known as the larger and smaller elliptical solar semidiurnal constituents.

In the expansion of the equilibrium tide, the diurnals M_1 and S_1 have zero amplitude because they are defined in the equatorial plane ($\delta = 0$), and the diurnal species are proportional to $\sin 2\delta$ (Eq. I.50). Nevertheless, the amplitudes of the sum and difference frequencies of Eq. I.62 are nonzero due to the changing declinations of the moon and sun. In the case of the moon, the declination changes with a period 27.3217 days (the sidereal month), i.e. with a speed of $0.549^\circ/\text{hour}$. The speeds associated with these changes are

$14.492 \pm 0.549^\circ/\text{hour} = 13.943$ and $15.041^\circ/\text{hour}$. These are the speeds of O_1 and K_1 . O_1 is the lunar declinational diurnal constituent, and K_1 , because as we shall see it also contains a contribution from the solar tidal potential.

The origin of the solar analogs of O_1 and K_1 is similar. In this case, the central frequency is $15^\circ/\text{hour}$ ($360^\circ/\text{msd}$). The period of the solar declination is 365.2564 days (the sidereal year), so the speed is $0.041^\circ/\text{hour}$. Hence, the sum and difference frequencies are $15. \pm 0.041^\circ/\text{hour} = 14.959$ and $15.041^\circ/\text{hour}$. The first is the speed of P_1 , and the latter is the same as we found for K_1 . P_1 is known as the solar declinational diurnal constituent.

Although as we have said there is no harmonic in the equilibrium tide with speed $15^\circ/\text{hour}$, there is a constituent with this frequency, known as S_1 , that is included in many tidal analyses. It owes its origin to meteorological effects and is commonly referred to as a "radiational tide".

7. NODAL PHASE

The moon's elliptical orbit around earth is at an angle to the earth's axis of rotation, and over time the orientation of the plane defined by the elliptical orbit rotates. As it does so, its nodes – the intersections of the orbit with the plane of the earth's equator – circuit westward through 360° of longitude. It does this once every 18.61 years. This regression of the lunar nodes or nodal cycle has a modulating effect on the amplitude and phase of all lunar tidal constituents, because over its course the maximum declination of the moon varies between $28^\circ 36'$ and $18^\circ 18'$ latitude north and south of earth's equator.

For each important line in the spectrum of the lunar equilibrium tide, there is a small sideband line, due to the nodal modulation, which cannot be resolved in typical tidal data (due to geophysical noise and limited duration). For example M_2 and N_2 , the two largest lunar constituents, each have significant sidebands whose frequency is less than the principal lines by an amount $2\pi/N$. The Doodson Numbers of the sidebands are identical to those of the principals except for the value of N , which equals one as opposed to zero (as is the case for M_2 and N_2). In both cases, the sideband signals are also out of phase from the principals by 180° .

The procedure normally adopted in tidal analysis was originally proposed by **Darwin (1901)**. Rather than attempt to resolve the small sideband frequencies due to the nodal modulation, the nodal cycle is accounted for by modulating the lunar and partly-lunar constituents with "corrections" known as the nodal factor $f_n(t)$ and nodal phase $u_n(t)$ (a subscript n is added here to emphasise that there are different factors and angles for different harmonics). The nodal factor is close to unity, and the phase angle is always small (they are identically unity and zero for purely solar terms). Hence, they amount to small, time-varying modulations of the principal lunar constituents.

We first write M_2 and its sideband, which we will call M_{2-} in the following form

$$\zeta_{M_2}^e \cos(2\omega t - 2s(t) + 2h(t))$$

$$\zeta_{M_{2-}}^e \cos(2\omega t - 2s(t) + 2h(t) - N(t) + 180^\circ)$$

These two waves are of the form $a \cos \omega t$ and $b \cos(\omega t + \theta(t))$ where $a \gg b$ and θ is a small angle. We wish to express their sum as a single cosine term of amplitude a modulated by $f(t)$, and with phase shift $u(t)$:

$$a \cos \omega t + b \cos(\omega t + \theta(t)) = f(t)a \cos(\omega t + u(t)) \tag{I.65}$$

Using a standard trigonometric identity and identifying like terms on either side of Eq.I.65, it follows that:

$$a + b \cos \theta(t) = f(t)a \cos u(t) \quad \text{and} \quad b \sin \theta(t) = f(t)a \sin u(t) \tag{I.66}$$

From the above equalities we obtain the following relations:

$$\tan u(t) = \frac{b \sin N(t)}{a + b \cos N(t)} \quad \text{and} \quad a^2 + 2ab + b^2 = f(t)^2 a^2 \tag{I.67}$$

Because amplitudes $a \gg b$, therefore the following approximations hold:

$$u(t) \simeq \tan u(t) = \frac{b}{a} \sin N(t) \quad \text{and} \quad f(t) = [1 + (\frac{b}{a})^2 + 2\frac{b}{a} \cos N(t)]^{1/2} \tag{I.68}$$

The regression of the lunar nodes also produces a small but measurable long-period constituent of the equilibrium tide, called the nodal tide, of period 18.61 years. Since the nodal sidebands (e.g. the one we called "M₂₋") contain a phase shift of 180°, when the nodal tide is a maximum, the modulation is a minimum. The nodal tide is often ignored as it is smaller than long-term oceanographic and geophysical changes. Its latitudinal dependence is the same as that of all the long period tides (cf. Eq.I.49): $\frac{3}{2} \cos^2 \phi - 1$. It is about 9 mm at the equator, falls to zero at 35° north and south, and reaches a maximum of 18 mm at the poles - about 13 mm when the elasticity of the earth is accounted for.

The f and u factors may be held constant for as much as a year without adding significant error to the prediction of a principle tidal constituent. Generally a value is calculated for the center of each prediction interval of up to one year. For predictions spanning longer than 18.6 years, the f and u factors may be eliminated, and terms included in the prediction to explicitly model the satellite sidebands as well as the 18.6 year nodal tide.

The nodal corrections are shown in Table I.6 for the important lunar and lunisolar constituents. Tabulations can also be found in [Shureman \(1958\)](#) and [Doodson and Warburg \(1941\)](#).

Table I.6 Nodal Parameters of the Major Tidal Constituents

Constituent	amplitude factor f	phase factor u
Mm	$1.000 - 0.130 \cos N(t)$	0.0°
Mf	$1.043 + 0.414 \cos N(t)$	$-23.7^\circ \sin N(t)$
O ₁ , Q ₁	$1.009 + 0.187 \cos N(t)$	$10.8^\circ \sin N(t)$
K ₁	$1.006 + 0.115 \cos N(t)$	$-8.9^\circ \sin N(t)$
N ₂ , M ₂	$1.000 - 0.037 \cos N(t)$	$-2.1^\circ \sin N(t)$
K ₂	$1.024 + 0.286 \cos N(t)$	$-17.7^\circ \sin N(t)$

In Fig. I.14, the time dependence of the amplitude factor (f) in the 20th century and beginning of 21st century is shown for M_2 , O_1 and K_1 . In 2006, the nodal cycle is acting to suppress M_2 while enhancing O_1 and K_1 .

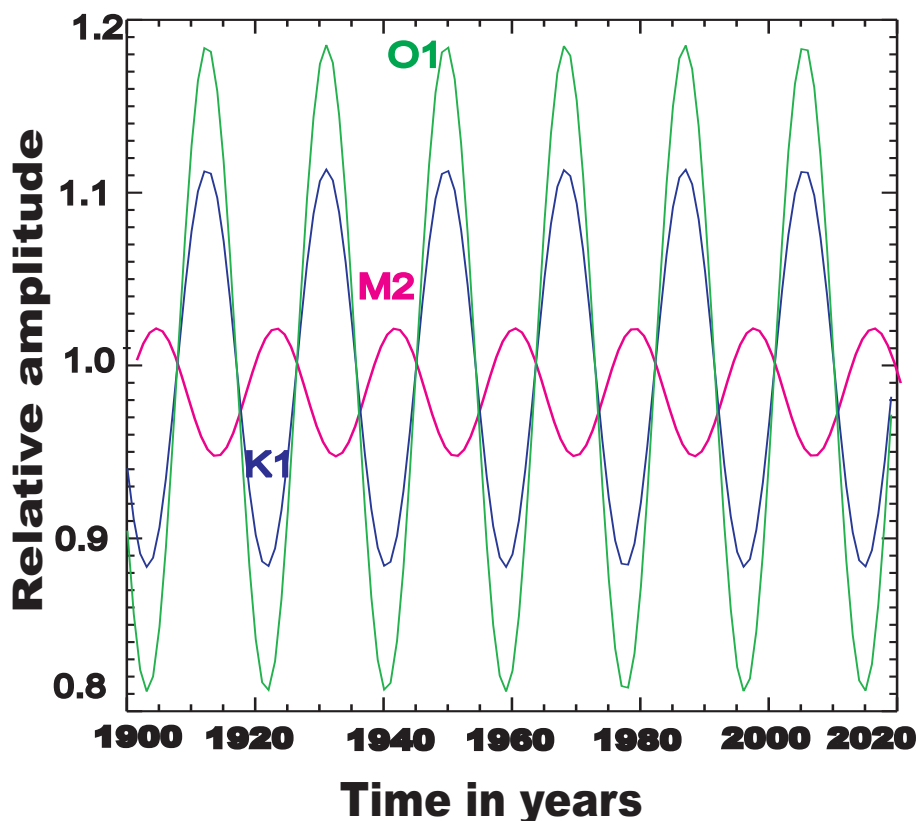


FIGURE I.14

Amplitude factor $f(t)$ for the M_2 , K_1 and O_1 constituents

8. LONGITUDE FORMULAS

Algebraic formulas for $s(t)$, $h(t)$, $p(t)$, $N(t)$, and $p_s(t)$ are given in a number of sources, including [Cartwright \(1982\)](#), [Doodson \(1921\)](#), [Franco \(1988\)](#), [Schureman \(1958\)](#) and the tidal package TASK-2000 (IOS, UK), and may be easily programmed. Several of these are given below, starting with Cartwright's. In each case, the results are phases valid at 0000 hours UT of the input day.

Cartwright's formula is based on an "epoch" of 1200 hours ET, 31 December 1899. Let d be the number of calendar days counted from the epoch, and $T=d/36525$. (A subroutine known as `zeller.for` is widely available for counting day numbers.) Noon on 1 January 1900 would be $d = 1$. Units are "revolutions", so must be multiplied by 360 to convert to degrees, for example. (The modulus would of course also be taken, eg. 2765.4° is equivalent to 245.4° .) Then:

$$s(t) = 0.751206 + 1336.855231 * T - 0.000003 * T^2$$

$$h(t) = 0.776935 + 100.002136 * T + 0.000001 * T^2$$

$$\begin{aligned}
p(t) &= 0.928693 + 11.302872 * T - 0.000029 * T^2 \\
N(t) &= 0.719954 - 5.372617 * T + 0.000006 * T^2 \\
p_s(t) &= 0.781169 + 0.004775 * T + 0.000001 * T^2
\end{aligned} \tag{I.69}$$

Phase increases linearly in time aside from a small quadratic factor which accounts for small secular trends in the speeds.

TASK-2000 is based on an epoch of 0000 hours UT, 1 January 1900. The formula is said to be accurate for at least the period 1800-2100:

$$\begin{aligned}
s(t) &= 277.0247 + 129.38481IY + 13.17639DL \\
h(t) &= 280.1895 - 0.23872IY + 0.98565DL \\
p(t) &= 334.3853 + 40.66249IY + 0.11140DL \\
N(t) &= 259.1568 - 19.32818IY - 0.05295DL \\
p_s(t) &= 281.2209 + 0.017192IY
\end{aligned} \tag{I.70}$$

In the above, IY = year -1900 (for example, for the year 1905, IY = 5, and for the year 1895, IY = -5), and DL = IL + IDAY -1, where IL is the number of leap years from 1900 (which was not a leap year) up to the start of year IY. Thus, IL = (IY-1)/4 in FORTRAN, and IDAY is the day number in the year in question. For example, for 12 January 1905, DL = 12. For years less than 1900 then one can compute IL = IABS(IY)/4 and DL = -IL + IDAY - 1 in FORTRAN.

Schureman (1958) is very similar to TASK-2000, using the same epoch, but with the Julian century (36526 days) as a unit of time.

$$\begin{aligned}
s(t) &= 277.0248 + 481267.8906T + 0.0011T^2 \\
h(t) &= 280.1895 + 36000.7689T + 3.0310^{-4}T^2 \\
p(t) &= 334.3853 + 4069.0340T - 0.0103T^2 \\
N(t) &= 259.1568 - 1934.142T + 0.0021T^2 \\
p_s(t) &= 281.2209 + 1.7192T + 0.00045T^2
\end{aligned} \tag{I.71}$$

The time T in the 19 century is

$$T = [365(Y - 1900) + (D - 1) + i]/36526$$

where i is the integer part of 0.25(Y-1901). The time is defined for 0000 hour UT on day D in year Y. Similar formula can be written for the 20 century time

$$T = [365(Y - 2000) + (D - 1) + i]/36526$$

Here i=INT[0.25(Y-2001)].

Franco's (1988) algorithms are given in terms of the Gregorian century, which may make them more convenient for dates further into the past or future. However, they carry one decimal place less on many terms than the foregoing algorithms, implying slightly less accuracy for the present time.

Three algorithms were given for computing reference phase, $V_n(t_0)$. $V_n(t_0)$ is computed as a function of the number of days elapsed between an epoch (e.g. 0000 hours UT on 1 January 1900) and the start of a day on which predictions are required. UT, or Universal Time, was the basis for the formulas of Doodson (1921) and Schureman (1976), while ET, or Ephemeris Time, was the basis for the formulas of Cartwright and Tayler (1971). At the start of the 1900s, the two time scales coincided. At the present time, however, they differ by about one minute. Estimates of $V_n(t_0)$ for a particular time in the present, using the two formulas, differ by an equivalent amount in degrees, depending on the speed of the constituent.

All of the above formulas are still in use at different institutions, and for most practical tide predictions the differences are negligible. However, Cartwright (1985) argues that compelling reasons exist for adopting new formulas which were published in the USA/UK Astronomical Almanac for 1984. These new formulas are based on the TDT time scale. TDT, which stands for Terrestrial Daylight Time, differs from UT by a varying amount, whose value can also be found in the almanac. The new formulas are also based on a more recent epoch: 1200 hours TDT on 31 December, 1999.

Where reference is found to GMT (Greenwich Mean Time), or Z (Zulu time), one can safely substitute in the term UT, which is essentially synonymous. However, data supplied by most observatories actually conform to a specific version of UT, such as UT1 or UTC. If accuracy greater than one second is required, then the version must be specified. The instantaneous difference between UTC and UT1 is available electronically.

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